

Supercongruences and Complex Multiplication

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Abstract

We study congruences involving truncated hypergeometric series of the form ${}_rF_{r-1}\left(\begin{smallmatrix} 1/2, \dots, 1/2 \\ 1, \dots, 1 \end{smallmatrix}; \lambda\right)_{(mp^s-1)/2} = \sum_{k=0}^{(mp^s-1)/2} ((1/2)_k/k!)^r \lambda^k$ where p is a prime and m, s, r are positive integers. These truncated hypergeometric series are related to the arithmetic of a family of algebraic varieties and exhibit Atkin and Swinnerton-Dyer type congruences. In particular, when $r = 3$, they are related to K3 surfaces. For special values of λ , with $s = 1$ and $r = 3$, our congruences are stronger than what can be predicted by the theory of formal groups because of the presence of elliptic curves with complex multiplications. They generalize a conjecture made by Rodriguez-Villegas for the $\lambda = 1$ case and confirm some other supercongruence conjectures at special values of λ .

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1. Introduction

The hypergeometric series ${}_rF_{r-1}$ is defined as

$${}_rF_{r-1}\left(\begin{smallmatrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_{r-1} \end{smallmatrix}; \lambda\right) := \sum_{k=0}^{\infty} \left(\frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{k! (b_1)_k (b_2)_k \cdots (b_{r-1})_k} \right) \lambda^k$$

where $(a)_k := a(a+1) \cdots (a+k-1)$ and none of the b_i is a negative integer [5]. The truncated hypergeometric series ${}_rF_{r-1}\left(\begin{smallmatrix} a_1, \dots, a_r \\ b_1, \dots, b_{r-1} \end{smallmatrix}; \lambda\right)_n$, is the degree n polynomial in λ obtained by truncating the hypergeometric series to the sum from $k = 0$ to n .

In this paper, we study the arithmetic of $F_r(\lambda)_n := {}_rF_{r-1}\left(\begin{smallmatrix} \frac{1}{2}, \dots, \frac{1}{2} \\ 1, \dots, 1 \end{smallmatrix}; \lambda\right)_n$; these values are related to the varieties $\mathcal{X}_r(\lambda) : W^2 = X_1 \cdots X_r (X_1 - X_2) \cdots (X_{r-1} - X_r) (X_r - \lambda X_1)$,

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which generalize the Legendre family of elliptic curves. In terms of counting points, Deuring's argument [10, pp. 255] yields that for any $\lambda \in \mathbb{F}_p$,

$$\#\mathcal{X}_r(\lambda)(\mathbb{F}_p) \equiv F_r(\lambda)_{p-1} \equiv F_r(\lambda)_{\frac{p-1}{2}} \pmod{p}.$$

In particular, when $r = 3$, $\mathcal{X}_3(\lambda)$ is a family of K3 surfaces that has been studied in [4, 18], denoted by $S_\lambda = \mathcal{X}_3(\lambda)$. Over an arbitrary finite field \mathbb{F} containing λ , Ahlgren, Ono, and Penniston showed that $\#(S_\lambda/\mathbb{F})$ can be computed using points on $E_\lambda : y^2 = (x-1)(x^2 - \frac{1}{1-\lambda})$ over \mathbb{F} [4]. By Dwork [12], for any $\lambda \in \mathbb{Z}_p$ such that $F_r(\lambda)_{p-1} \not\equiv 0 \pmod{p}$ (i.e. p ordinary for $\mathcal{X}_r(\lambda)$) and for any integer $m \geq 1$,

$$F_r(\lambda)_{mp^s-1} \equiv \gamma(\lambda) F_r(\lambda^p)_{mp^{s-1}-1} \pmod{p^s} \quad (1)$$

for a p -adic unit $\gamma(\lambda)$ which is independent of m , but may vary if λ is replaced by λ^p .

Theorem 1. *Let p be an odd prime, $\lambda \in \mathbb{Z}_p$ such that $\mathcal{X}_r(\lambda)$ has good ordinary reduction at p . Then $\ell(\tau) = \sum_{n \geq 1} \frac{F_r(\lambda)_n}{2n+1} \tau^{2n+1}$ is the logarithm of a formal group over \mathbb{Z}_p , which is isomorphic to a formal group attached to $\mathcal{X}_r(\lambda)$ constructed by Stienstra [26]. When $r = 3$, for all integers $s \geq 1$ and m odd*

$$F_3(\lambda)_{\frac{mp^s-1}{2}} \equiv \left(\frac{\lambda-1}{p} \right) \alpha_{p,\lambda}^2 F_3(\lambda)_{\frac{mp^{s-1}-1}{2}} \pmod{p^s},$$

with $\alpha_{p,\lambda}$ being the unit root of $X^2 - [p+1 - \#(E_\lambda/\mathbb{F}_p)]X + p = 0$.

At special values of λ such that E_λ has complex multiplications (CM), stronger congruences have been observed. These congruences are known as *supercongruences*. Rodriguez-Villegas conjectured several supercongruences involving truncated hypergeometric series in [24], including the following: for odd primes p ,

$$F_3(1)_{\frac{p-1}{2}} = {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1 \right)_{\frac{p-1}{2}} \equiv b_p \pmod{p^2}$$

where b_p is the p th coefficient of the weight 3 cusp form $\eta(4z)^6$, where $\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ with $q = e^{2\pi iz}$, is the eta function. The $\lambda = 1$ case was proved by Van Hamme in [32, 1996] and by Ono in [23, 1998], using different methods.

Similarly, Z.-W. Sun conjectured (see remark 1.4 in [29]) a congruence for the $\lambda = 64$ case:

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 64 \right)_{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 (64)^k \equiv a_p \pmod{p^2}$$

where $a_p = 0$ if $p \equiv 3, 5, 6 \pmod{7}$ and $a_p = 4x^2 - 2p$ where $p = x^2 + 7y^2$, $x, y \in \mathbb{Z}$, if $p \equiv 1, 2, 4 \pmod{7}$. In fact, this a_p is just the p th coefficient of $\eta(z)^3 \eta(7z)^3$.

Theorem 2. *Let $\lambda \neq 1$ be an algebraic number such that E_λ has complex multiplications. Let p be a prime and E_λ have a model defined over \mathbb{Z}_p with good reduction modulo $p\mathbb{Z}_p$. Then*

$${}_3F_2 \left(\begin{matrix} 1/2, & 1/2, & 1/2 \\ & 1 & 1 \end{matrix} ; \lambda \right)_{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 \lambda^k \equiv \left(\frac{\lambda-1}{p} \right) \alpha_{p,\lambda}^2 \pmod{p^2}$$

where $\alpha_{p,\lambda}$ is the unit root of $X^2 - [p+1 - \#(E_\lambda/\mathbb{F}_p)]X + p = 0$ if E_λ is ordinary at p ; and $\alpha_{p,\lambda} = 0$ if E_λ is supersingular at p .

Our result confirms the Conjecture of Sun mentioned above. We conjecture that the statement of Theorem 1 is true modulo p^{2s} when E_λ has CM.

The hypergeometric series ${}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} ; \lambda \right)$, when *not* truncated, gives an expression for the real period of the elliptic curve E_λ as McCarthy shows in [20].

We derive the following corollary to Theorem 2 in section §4:

Corollary 3. *Let H_k be the harmonic sum $\sum_{j=1}^k \frac{1}{j}$. If E_λ is a CM elliptic curve, then for almost all primes p such that λ embeds in \mathbb{Z}_p ,*

$$\sum_{i=0}^{\frac{p-1}{2}} \binom{2i}{i}^3 \left(\frac{\lambda}{64} \right)^i \left(6(H_{2i} - H_i) + \left(\frac{(\frac{\lambda}{64})^{p-1} - 1}{p} \right) \right) \equiv 0 \pmod{p}.$$

Below is one simple, special case of these congruences for $\lambda = 64$.

Corollary 4. *For all primes $p > 3$, we have*

$$\sum_{i=1}^{\frac{p-1}{2}} \binom{2i}{i}^3 \sum_{j=1}^i \frac{1}{i+j} \equiv 0 \pmod{p}. \quad (2)$$

In general, such congruences are difficult to prove. For similar work, see [1, 2]. Remark 1 of [19] reduces an open supercongruence to a congruence like (2).

We end our introduction with another motivation for supercongruences. It is known that the coefficients of weight- k noncongruence modular forms satisfy the so-called Atkin and Swinnerton-Dyer congruences [6, 25]. These congruences are supercongruences if $k > 2$ [25] and have played an important role in understanding the characterizations of genuine noncongruence modular forms [17].

2. Atkin and Swinnerton-Dyer congruences of a family of truncated hypergeometric series

The Hasse invariants of the Legendre family of elliptic curves $L_\lambda : y^2 = x(x-1)(x-\lambda)$ are $A_p(\lambda) = (-1)^{(p-1)/2} \sum_{i=0}^{(p-1)/2} \left(\frac{p-1}{2}\right)_i^2 \lambda^i$ and

$$A_p(\lambda) \equiv [p+1 - \#L_\lambda(\mathbb{F}_p)] \equiv (-1)^{(p-1)/2} {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; \lambda\right)_{(p-1)/2} \pmod{p}$$

where ${}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; \lambda\right)$ is the unique-up-to-scalar, holomorphic-near-0 solution of the Picard-Fuchs equation of L_λ (see [8]). In [12], Dwork proved that when $A_p(\lambda) \not\equiv 0 \pmod{p}$, i.e. when L_λ is ordinary at p ,

$$\frac{{}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; \lambda\right)_{p^s-1}}{{}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; \lambda^p\right)_{p^{s-1}-1}} \equiv \gamma_p(\lambda) \pmod{p^s}$$

where $\gamma_p(\lambda)$ is a p -adic unit which usually varies when λ is replaced by λ^p . If $\lambda = \lambda^p$, the limit is $\left(\frac{-1}{p}\right) \beta_p$, where β_p is the p -adic unit root of $T^2 - [p+1 - \#(L_\lambda/\mathbb{F}_p)]T + p = 0$. When L_λ has CM, the γ_p can often be obtained via Gauss sums and Jacobi sums. Yu has further extended Dwork's results to Dwork families of algebraic varieties [31].

We now compare Dwork's result with what can be predicted from Atkin and Swinnerton-Dyer congruences. Let $P_n(x)$ denote the n th Legendre polynomial, which can be defined by $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$ [5, 9, 28]. These polynomials form an important class of orthogonal polynomials and have several nice properties; but the fact most relevant to our application is that they have generating function $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$. Because of this, special values of $P_n(x)$ show up in certain expansions of differential forms on elliptic curves. The first few Legendre Polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

For elliptic curves of the form $\mathcal{E} : y^2 = x(x^2 + Ax + B)$ defined over \mathbb{Z}_p with $t = x/y$ as a local parameter at the point at infinity (where t has a simple zero), Coster and van Hamme showed that the coefficients of the t -expansion of the invariant differential form $-\frac{dx}{2y}$ of \mathcal{E} are essentially just special values of Legendre Polynomials (see formula (1) of [9]). In particular, for $L_\lambda : y^2 = x(x-1)(x-\lambda)$,

$$-\frac{dx}{2y} = \sum_{k=0}^{\infty} P_k\left(\frac{1+\lambda}{1-\lambda}\right) (\lambda-1)^k t^{2k+1} \frac{dt}{t}. \quad (3)$$

For $k \geq 0$, letting $a_{2k+1} = P_k\left(\frac{1+\lambda}{1-\lambda}\right) (\lambda-1)^k$ and using formulas (5) and (6) in [33], we have

$$a_{2k+1} = {}_2F_1\left(\begin{matrix} -k, -k \\ 1 \end{matrix}; \lambda\right) (-1)^k = {}_2F_1\left(\begin{matrix} -k, 1+k \\ 1 \end{matrix}; \frac{-\lambda}{1-\lambda}\right) (\lambda-1)^k. \quad (4)$$

Note that these are terminating hypergeometric series; i.e. degree k polynomials of λ , because of the $-k$ argument.

The Atkin and Swinnerton-Dyer congruences (ASD) for elliptic curves (Theorem 4 of [6]) imply that if λ embeds in \mathbb{Q}_p and L_λ has good reduction modulo p , then for all positive integers m, s ,

$$a_{mp^{s+1}} - A_p a_{mp^s} + p a_{mp^{s-1}} \equiv 0 \pmod{p^{s+1}} \quad (5)$$

where $A_p = p + 1 - \#(L_\lambda/\mathbb{F}_p)$. We define a_k to be 0 if k is not integral, as may happen for the final term if $s = 0$. The factors of $(-1)^k$ and $(\lambda - 1)^k$ can be omitted from the expressions (4) for a_k if we adjust the middle coefficient of the ASD congruence by the Legendre symbols $\left(\frac{-1}{p}\right)$ or $\left(\frac{\lambda-1}{p}\right)$, respectively.

Essentially, the ASD congruences say that for fixed p and m , terms of the sequence $\{a_{mp^s}\}$ satisfy a three-term congruence with increasing p -adic precision as s increases. The ASD congruences generalize the Hecke recursion: Fourier coefficients a_n of weight $k = 2$, normalized Hecke newforms with trivial nebentypus satisfy the three-term recursion, for all $m, s \geq 1$ and all p ,

$$a_{mp^{s+1}} - a_p a_{mp^s} + p a_{mp^{s-1}} = 0. \quad (6)$$

In the ASD congruences for an elliptic curve \mathcal{E} , we distinguish two cases. If the middle coefficient A_p is divisible by p , we say that \mathcal{E} is *supersingular* at p or simply that p is supersingular. Otherwise, we say \mathcal{E} is *ordinary* at p or that p is ordinary. Dwork's congruences, in which consecutive ratios of certain terms in a sequence converge to a p -adic limit, are related to ASD congruences at ordinary primes. At ordinary primes, let $\beta_{p,\lambda}$ be the p -adic unit root of $T^2 - [p + 1 - \#(L_\lambda/\mathbb{F}_p)]T + p$. Then the ASD congruences imply that $\frac{a_{p^s}}{a_{p^{s-1}}} \equiv \beta_{p,\lambda} \pmod{p^s}$. Note that $a_1 = 1$, so the $s = 1$ case of this congruence is just $a_p \equiv \beta_{p,\lambda} \pmod{p}$.

Thus, at all ordinary primes p of L_λ , we obtain congruences for hypergeometric functions using the expansions given in (4) with $k = \frac{p^s-1}{2}$:

$$\frac{{}_2F_1\left(\frac{1-p^s}{2}, \frac{1 \pm p^s}{2}; \lambda\right)}{{}_2F_1\left(\frac{1-p^{s-1}}{2}, \frac{1 \pm p^{s-1}}{2}; \lambda\right)} \equiv \left(\frac{-1}{p}\right) \beta_{p,\lambda} \pmod{p^s}. \quad (7)$$

The twist by the character $\left(\frac{-1}{p}\right)$ accounts for the factor $(-1)^k$ in the first equality in (4) and for the change of argument from $\frac{-\lambda}{1-\lambda}$ to λ and the factor $(\lambda - 1)^k$ in the second equality in (4).¹

¹The curve $L_{\frac{-\lambda}{1-\lambda}}$ is isomorphic to the twist of L_λ by $\left(\frac{1-\lambda}{p}\right)$. Combining this with the factor of $(\lambda - 1)^k$, we get a twist by $\left(\frac{-1}{p}\right)$.

Both the Hecke recursion and ASD congruences are related to formal groups. Any sequence of p -adic integers a_n with a_1 being a p -adic unit satisfying the congruences (5) can be used to construct a formal group law $F(x, y) := \ell^{-1}(\ell(x) + \ell(y)) \in \mathbb{Z}_p[[x, y]]$ with formal logarithm $\ell(x) = \sum_{n=1}^{\infty} \frac{a_n}{n} x^n$. Note that the power series $\ell(x)$ and $\ell^{-1}(x)$ have denominators with arbitrarily large powers of p in general; it is the congruences (5) that guarantee that the composition $F(x, y)$ has no denominators of p . Details can be found in [11] or other formal group theory references; specifically, the p -typification of $\ell^{-1}(\ell(x) + \ell(y))$ is isomorphic over \mathbb{Z}_p to a formal group law with F_p -type $F_p = \{A_p\} - V_p$, where F_p is the p th Frobenius operator, $\{A_p\}$ is a Hilbert operator, and V_p is the p th Verschiebung operator in the Cartier-Dieudonné module. (This integer A_p corresponds to the eigenvalue of the p th Hecke operator; recall that the Hecke operator is just the sum of the Frobenius and Verschiebung operators on the space of congruence cusp forms.)

In fact, in the ordinary case, when $a_p \not\equiv 0 \pmod{p}$, our formal group is isomorphic to one with F_p -type $F_p = \{\beta_p\}$, where β_p is a p -adic unit. This fact corresponds to the congruences $a_{mp^s} \equiv \beta_p a_{mp^{s-1}} \pmod{p^s}$, which account for the many congruences we consider for expressions of the form $\frac{a_{p^s}}{a_{p^{s-1}}}$. Dwork's congruences, in which the denominator involves λ^p instead of λ , correspond to a formal group law over $\mathbb{Z}_p[\lambda]$, which is isomorphic over $\mathbb{Z}_p[[\lambda]]$ to a formal group law with F_p -type $F_p = \{\gamma_p(\lambda)\}$, for some $\gamma_p(\lambda) \in \mathbb{Z}_p[[\lambda]]$. Dwork's formal group law can be specialized to many different formal group laws over \mathbb{Z}_p by choosing suitable $\lambda \in \mathbb{Z}_p$ (as we show in Proposition 5), but the F_p -type cannot be specialized by substituting the value of λ in $\gamma_p(\lambda)$, because the Hilbert structures on $\mathbb{Z}_p[[\lambda]]$ and on \mathbb{Z}_p are incompatible.² So, even though the many specializations of Dwork's formal groups are all isomorphic over \mathbb{Z}_p as long as $\lambda \pmod{p}$ is fixed, Dwork's congruences give different limits $\gamma_p(\lambda)$ for these λ . When we omit the p th power from the denominator, we obtain the same limit $(\frac{-1}{p})\beta_p$ for all λ with $\lambda \pmod{p}$ fixed. Note that $\gamma_p(\lambda) = (\frac{-1}{p})\beta_p$ if $\lambda = \lambda^p$.

The perspective of formal groups motivates our approach to congruences and supercongruences of hypergeometric functions; many of the congruences found in the literature seem to be initial cases of the ASD congruence structure. In fact, our second theorem appears to be the very first case of an ASD congruence; in Conjecture 9, we suggest that infinitely many more congruences hold.

$$\text{Recall } F_r(\lambda)_n := {}_rF_{r-1} \left(\begin{matrix} 1/2, & 1/2, & \cdots, & 1/2 \\ & 1, & \cdots, & 1 \end{matrix} ; \lambda \right)_n.$$

²Dwork's congruences make use of the endomorphism τ of $\mathbb{Z}_p[[\lambda]]$ sending $f(\lambda)$ to $f(\lambda^p)$. This endomorphism satisfies $\tau(f(\lambda)) = f(\lambda^p) \equiv f(\lambda)^p \pmod{p}$ for all $f(\lambda) \in \mathbb{Z}_p[\lambda]$ and is a ring endomorphism, so it gives a Hilbert structure to formal group laws over $\mathbb{Z}_p[[\lambda]]$. On \mathbb{Z}_p , however, we must use the identity endomorphism ι , which also satisfies $\iota(x) = x \equiv x^p \pmod{p}$ for all $x \in \mathbb{Z}_p$. This is not compatible with a specialization of τ unless we choose λ satisfying $\lambda = \lambda^p$.

Proposition 5. *Let p be an odd prime and $A(n)$ be a sequence of numbers in \mathbb{Z}_p such that $A(0)$ is a unit, and suppose the following three conditions hold.*

a) *For all $n, m, s \in \mathbb{Z}_+$,*

$$\frac{A(n + mp^{s+1})}{A(\lfloor \frac{n}{p} \rfloor + mp^s)} \equiv \frac{A(n)}{A(\lfloor \frac{n}{p} \rfloor)} \pmod{p^{s+1}}.$$

b) *For all $n \in \mathbb{Z}_+$, $A(n)/A(\lfloor \frac{n}{p} \rfloor) \in \mathbb{Z}_p$.*

c) *$A(i) \equiv 0 \pmod{p}$ for all $\frac{p+1}{2} \leq i < p$.*

Then for any $x \in \mathbb{Z}_p$ such that $\sum_{i=0}^{p-1} A(i)x^i \not\equiv 0 \pmod{p}$, there exists a p -adic unit α , depending only on $x \pmod{p}$, such that for any odd integer m

$$\frac{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i}{\sum_{i=0}^{\frac{mp^{s-1}-1}{2}} A(i)x^i} \equiv \alpha \pmod{p^{s-d_m}}$$

where $d_m = \max_{s \geq 0} \left(\text{ord}_p \left(\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i \right) \right)$.

The proof implies that d_m is finite, unless $\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i = 0$ for all s . Also, if $\sum_{i=0}^{\frac{mp-1}{2}} A(i)x^i \not\equiv 0 \pmod{p}$, then $d_m = 0$; note that $d_1 = 0$.

PROOF. Compare to Theorem 2 of Dwork; we assume $A(n) = B(n)$ (see Dwork's notation) and we add condition c), which will allow us to go use sums to $\frac{mp^s-1}{2}$ instead of to $mp^s - 1$. Let $F(x) = \sum_{i \geq 0} A(i)x^i$. Taking the sum of congruence (2.1) in [12] from $m = 0$ to $m = n$, we have

$$F(x) \sum_{j=0}^{(n+1)p^s-1} A(j)x^{pj} \equiv F(x^p) \sum_{j=0}^{(n+1)p^{s+1}-1} A(j)x^j \pmod{p^{s+1}\mathbb{Z}_p[[x]]}.$$

Under the additional assumption c) and the argument of Dwork (Theorem 2), we can obtain that for all integers $n, s \geq 0$

$$F(x) \sum_{j=np^s+\frac{p^s+1}{2}}^{(n+1)p^s-1} A(j)x^{pj} \equiv F(x^p) \sum_{j=np^s+\frac{p^{s+1}+1}{2}}^{(n+1)p^{s+1}-1} A(j)x^j \pmod{p^{s+1}\mathbb{Z}_p[[x]]}.$$

(For full details and notation, please see [12].) If we subtract the congruences above, we obtain

$$F(x) \sum_{j=0}^{\frac{mp^s-1}{2}} A(j)x^{pj} \equiv F(x^p) \sum_{j=0}^{\frac{mp^{s+1}-1}{2}} A(j)x^j \pmod{p^{s+1}\mathbb{Z}_p[[x]]} \quad (8)$$

where $m = 2n + 1$ could be any odd number.

For fixed m , we divide congruence (8) for consecutive s , obtaining

$$\frac{\sum_{i=0}^{\frac{mp^{s+1}-1}{2}} A(i)x^i}{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i} \equiv \frac{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^{ip}}{\sum_{i=0}^{\frac{mp^{s-1}-1}{2}} A(i)x^{ip}} \pmod{p^{s-d_m}\mathbb{Z}_p[[x]]}. \quad (9)$$

Consequently, the left-hand side, when viewed as a formal power series of the form $\sum_{n \geq 0} a_{n,s+1}x^n$, satisfies that $p^{s-d_m} \mid a_{n,s+1}$ if $p \nmid n$. Iterating this idea, we have

$$\frac{\sum_{i=0}^{\frac{mp^{s+1}-1}{2}} A(i)x^i}{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i} \equiv \frac{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^{ip}}{\sum_{i=0}^{\frac{mp^{s-1}-1}{2}} A(i)x^{ip}} \equiv \frac{\sum_{i=0}^{\frac{mp^{s-1}-1}{2}} A(i)x^{ip^2}}{\sum_{i=0}^{\frac{mp^{s-2}-1}{2}} A(i)x^{ip^2}} \pmod{p^{s-1-d_m}\mathbb{Z}_p[[x]]}.$$

So $p^{s-1-d_m} \mid a_{n,s+1}$ if $p^2 \nmid n$. By induction, $p^{s-i-d_m} \mid a_{n,s+1}$ if $p^{1+i} \nmid n$.

Using this, we show that the left hand side of (9), for fixed m and s , is determined modulo p^{s+1-d_m} by $\lambda \pmod{p}$, provided that $\sum_{i=0}^{p-1} A(i)x^i \not\equiv 0 \pmod{p}$. Note that condition b) then implies that $\sum_{i=0}^{p^s-1} A(i)x^i$ is a p -adic unit for each s . We show that

$$\frac{\sum_{i=0}^{\frac{mp^{s+1}-1}{2}} A(i)x^i}{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i} - \frac{\sum_{i=0}^{\frac{mp^{s+1}-1}{2}} A(i)(x+pk)^i}{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)(x+pk)^i} = \sum_{n \geq 0} a_{n,s+1}(x^n - (x+pk)^n)$$

is congruent to 0 modulo p^{s+1-d_m} for any $k \in \mathbb{Z}_p$. Writing $n = p^e n'$ with $p \nmid n'$, then $p^{s-e-d_m} \mid a_{n,s+1}$ by the paragraph above, and $x^n - (x+pk)^n \equiv 0 \pmod{p^{e+1}}$ so $p^{s+1-d_m} \mid a_{n,s+1}(x^n - (x+pk)^n)$ term by term.

Since $x \equiv x^p \pmod{p}$, for fixed x such that $\sum_{i=0}^{p-1} A(i)x^i \not\equiv 0 \pmod{p}$ we can replace congruence (9) with

$$\frac{\sum_{i=0}^{\frac{mp^{s+1}-1}{2}} A(i)x^i}{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i} \equiv \frac{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i}{\sum_{i=0}^{\frac{mp^{s-1}-1}{2}} A(i)x^i} \pmod{p^{s-d_m}}; \quad (10)$$

thus there is some $\alpha \in \mathbb{Z}_p$ such that $\frac{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i}{\sum_{i=0}^{\frac{mp^{s-1}-1}{2}} A(i)x^i} \equiv \alpha \pmod{p^{s-d_m}}$.

To see that this limit α is independent of m as long as $x \pmod{p}$ remains fixed, we rearrange congruence (8) to obtain, for any odd m ,

$$\frac{F(x)}{F(x^p)} \equiv \frac{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)x^i}{\sum_{i=0}^{\frac{mp^{s-1}-1}{2}} A(i)x^{ip}} \equiv \frac{\sum_{i=0}^{\frac{p^s-1}{2}} A(i)x^i}{\sum_{i=0}^{\frac{p^{s-1}-1}{2}} A(i)x^{ip}} \pmod{p^{s-d_m}\mathbb{Z}_p[[x]]}.$$

We may choose \hat{x} to be the Teichmüller lift of x , so that $\hat{x} \equiv x \pmod{p}$ and $\hat{x}^p = \hat{x}$. Then $\frac{\sum_{i=0}^{\frac{mp^s-1}{2}} A(i)\hat{x}^i}{\sum_{i=0}^{\frac{mp^{s-1}-1}{2}} A(i)\hat{x}^i} \equiv \frac{\sum_{i=0}^{\frac{p^s-1}{2}} A(i)\hat{x}^i}{\sum_{i=0}^{\frac{p^{s-1}-1}{2}} A(i)\hat{x}^i} \equiv \alpha \pmod{p^{s-d_m}}$. Thus, the limit α is independent of m .

Now take $A(n) = \left(\frac{(\frac{1}{2})_n}{n!}\right)^r$ for any integer $r \geq 1$. It follows from [12], a), b) and c) of the above hold. Then $\sum_{j=0}^n A(j)x^j = F_r(\lambda)_n$.

Theorem 6. *For any positive integer r , odd prime p , and $\lambda \in \mathbb{Z}_p$ such that $F_r(\lambda)_{\frac{p-1}{2}} \not\equiv 0 \pmod{p}$, there is a p -adic unit α_r , such that for all integers $s \geq 1$ and m odd*

$$\frac{F_r(\lambda)_{\frac{mp^s-1}{2}}}{F_r(\lambda)_{\frac{mp^{s-1}-1}{2}}} \equiv \alpha_r \pmod{p^{s-d_m}}.$$

Moreover, the formal group with logarithm $\sum_{n \geq 0} \frac{F_r(\lambda)_n}{2n+1} \tau^{2n+1}$ is isomorphic over \mathbb{Z}_p to the formal group $H^{r-1}(\mathcal{X}_r, \hat{G}_{m, \mathcal{X}_r})$ where $\mathcal{X}_r(\lambda) : W^2 = X_1 X_2 \cdots X_r (X_1 - X_2)(X_2 - X_3) \cdots (X_r - \lambda X_1)$.

PROOF. From the previous proposition, we know the first claim holds. Let $\ell(\tau) = \sum_{n \geq 0} \frac{F_r(\lambda)_n}{2n+1} \tau^{2n+1}$. It follows that the formal group law $\ell^{-1}(\ell(x) + \ell(y))$ is integral at p .

By Theorem 2 of [26], there is a 1-dimensional formal group $H^{r-1}(\mathcal{X}_r, \hat{G}_{m, \mathcal{X}_r})$ with logarithm $\sum \frac{b_{r,n}(\lambda)}{2n+1} \tau^{2n+1}$ where $b_{r,n}(\lambda) = {}_r F_{r-1} \left(\begin{matrix} -n, & \dots, & -n \\ 1, & \dots, & 1 \end{matrix} ; (-1)^r \lambda \right)$. By an observation of Koblitz [15], we know

$$\lim_{s \rightarrow \infty} \frac{b_{r, \hat{\lambda}, (mp^s-1)/2}}{b_{r, \hat{\lambda}, (mp^{s-1}-1)/2}} = \lim_{s \rightarrow \infty} \frac{a_{r, \hat{\lambda}, (mp^s-1)/2}}{a_{r, \hat{\lambda}, (mp^{s-1}-1)/2}} = \alpha_r \in \mathbb{Z}_p,$$

which implies that both formal groups are isomorphic over \mathbb{Z}_p to a formal group law with F_p -type $F_p = \{\alpha_r\}$, and thus are isomorphic to each other over \mathbb{Z}_p .

(When $r = 3, \lambda = 1$, these formal groups are also isomorphic to a formal Brauer group constructed in [27].)

In fact, we expect this formal group law to be integral at every prime except 2 and all prime places \mathfrak{p} such that $\text{val}_{\mathfrak{p}}(\lambda) < 0$, but the integrality at all ordinary and inert primes follows immediately from Theorem 6.

3. Supercongruences

Any formal group law with logarithm $\ell(x) = \sum_{n \geq 1} \frac{a_n}{n} x^n$ that is integral at p and that has finite F_p -type satisfies an infinite family of congruences, which express a_{mp^s} as a linear combination of lower-index coefficients a_n modulo p^s . Many of the congruences for hypergeometric series can be seen from this perspective; however, the congruences are often much stronger, giving a formula for a_{mp^s} modulo p^{2s} , p^{3s} , or even p^{4s} or more. Such congruences, that are stronger than what are predicted by the existence of a formal group, are called *supercongruences*. One source of supercongruences are ASD congruences of Fourier coefficients of cusp forms with weight $k > 2$; these exhibit congruences of order $p^{(k-1)s}$ [25]. We are interested in another source of supercongruences: extra symmetries of the underlying variety, such as complex multiplications of the elliptic curves.

Here is a well-known example of a supercongruence. Beukers conjectured that for all odd primes p

$${}_4F_3 \left(\begin{matrix} \frac{1-p}{2}, \frac{1-p}{2}, \frac{1+p}{2}, \frac{1+p}{2} \\ 1, 1, 1 \end{matrix} ; 1 \right) \equiv c_p \pmod{p^2}$$

where the left hand side is the $\frac{p-1}{2}$ -th Apéry number $\sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k}^2 \binom{(p-1)/2+k}{k}^2$ and c_p is the p th coefficient of the weight-4 modular form $\eta(2z)^4 \eta(4z)^4$; this was proved by Ahlgren and Ono [2]. Ahlgren and Ono's approach uses Gaussian hypergeometric functions (see [2] and [22, Chapter 11]) and has inspired much later work including Kilbourn's result ([16]) that for all primes $p > 2$

$${}_4F_3 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} ; 1 \right)_{(p-1)/2} \equiv c_p \pmod{p^3}.$$

We would like to understand when we should expect supercongruence for our truncated hypergeometric series and have a better understanding for $r = 2$ and 3 cases by using the following theorem of Coster and van Hamme.

Theorem 7 (Coster and van Hamme, [9]). *Let p be an odd prime and d a square-free positive integer such that $(\frac{-d}{p}) = 1$. Let K be an algebraic number field such that $\sqrt{-d} \in K$ and $K \subset \mathbb{Q}_p$. Consider the elliptic curve*

$$\mathcal{E} : Y^2 = X(X^2 + AX + B)$$

with $A, B \in K$, where A and $\Delta = A^2 - 4B$ are p -adic units. Let ω and ω' be a basis of periods of \mathcal{E} and suppose that $\tau = \omega'/\omega \in \mathbb{Q}(\sqrt{-d})$, τ has positive imaginary part. Let $\pi, \bar{\pi} \in \mathbb{Q}(\sqrt{-d})$ be complex conjugates such that $\pi\bar{\pi} = p$, with $\bar{\pi}$ a p -adic unit, $\pi = u_1 + v_1\tau$, and $\pi\tau = u_2 + v_2\tau$ with u_1, v_1, u_2, v_2 integers and v_1 even. Then we have

$$P_{\frac{mp^r-1}{2}} \left(\frac{A}{\sqrt{\Delta}} \right) \equiv \varepsilon^{mp^{r-1}} \cdot \bar{\pi} \cdot P_{\frac{mp^{r-1}-1}{2}} \left(\frac{A}{\sqrt{\Delta}} \right) \pmod{\pi^{2r}}, \quad (11)$$

where m and r are positive integers, with m odd, and $\varepsilon = i^{-u_2 v_2 + v_2 + p - 2}$, where $P_n(x)$ is the n th Legendre polynomial.

Note that the assumptions imply that \mathcal{E} has CM, $A = 3\wp(\frac{1}{2}\omega)$, $\sqrt{\Delta} = \wp(\frac{1}{2}\omega' + \frac{1}{2}\omega) - \wp(\frac{1}{2}\omega')$, where $\wp(z)$ is the Weierstrass \wp -function. The technical details such as A and D being p -adically integral are always satisfied in our cases, and $\varepsilon^{mp^{r-1}}$ is just a quartic character, which we can explicitly identify. The main point of the theorem is the existence of supercongruences arising from an elliptic curve \mathcal{E} with CM. While Coster and van Hamme interpreted the congruence as inclusion in an ideal of the ring of integers of K , we interpret all of our congruences p -adically and simply replace $(\bmod \pi^{2r})$ with $(\bmod p^{2r})$.

Thus, for CM curves L_λ , we can double the strength of the congruences (7). The factor $(\lambda - 1)^k$ from the formulas (4) and (3) only shows up as a quadratic character in the congruences (7), because $(\lambda - 1)^{\frac{mp^s - mp^{s-1}}{2}} \equiv \left(\frac{\lambda - 1}{p}\right) \pmod{p^s}$; however for supercongruences modulo p^{2s} , it must be included. If λ is a CM value for the family E_λ , then at ordinary primes p , for all $m, s \geq 1$ with m odd, we have

$$\begin{aligned} & {}_2F_1\left(\frac{1-mp^s}{2}, \frac{1-mp^s}{2}; \lambda\right) \\ & \equiv \left(\frac{1-\lambda}{p}\right) (\lambda - 1)^{\frac{mp^s - mp^{s-1}}{2}} \beta_{p,\lambda} {}_2F_1\left(\frac{1-mp^{s-1}}{2}, \frac{1-mp^{s-1}}{2}; \lambda\right) \pmod{p^{2s}} \end{aligned}$$

and

$${}_2F_1\left(\frac{1-mp^s}{2}, \frac{1+mp^s}{2}; \lambda\right) \equiv \left(\frac{-1}{p}\right) \beta_{p,\lambda} {}_2F_1\left(\frac{1-mp^{s-1}}{2}, \frac{1+mp^{s-1}}{2}; \lambda\right) \pmod{p^{2s}}.$$

When $m = s = 1$, we have

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \left(\frac{\lambda}{16}\right)^k & \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2} + k}{k} \binom{\frac{p-1}{2}}{k} (-\lambda)^k \pmod{p^2} \\ & \equiv \left(\frac{\lambda - 1}{p}\right) \left(\frac{1}{\lambda - 1}\right)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \lambda^k \pmod{p^2}. \end{aligned}$$

These generalize the supercongruences in [7] with $\lambda = 2$ and as well as one of Mortenson's supercongruences with $\lambda = 1$, in which L_λ degenerates [21].

Theorem 2 relates values of the truncated hypergeometric function $F_3(\lambda)_n$ to the family of K3 surfaces $S_\lambda : z^2 = xy(x-1)(y-1)(x-\lambda y)$ and the family of elliptic curves $E_\lambda : y^2 = (x-1)(x^2 - \frac{1}{1-\lambda})$. Let $A_p(\lambda) = \#S_\lambda(\mathbb{F}_p) - p^2 - 1$. Then $F_3(\lambda)_{\frac{p-1}{2}} = A_p(\lambda) \pmod{p}$. The variation of the complex structure of the family S_λ of K3 surfaces is again depicted by its Picard-Fuchs differential equation, which is projectively equivalent to

the symmetric square of the Picard-Fuchs equation of $E_\lambda : y^2 = (x-1)(x^2 - \frac{1}{1-\lambda})$ [18]. In terms of arithmetic, if we let $a_p(\lambda) = p+1 - \#E_\lambda(\mathbb{F}_p)$, then $A_p(\lambda) = \left(\frac{1-\lambda}{p}\right) (a_p(\lambda)^2 - p)$ [4].

Specializing formula (1) of [9] to E_λ and using uniformizer $t = \frac{x-1}{y}$ as a local parameter at the point at infinity, we obtain the expansion

$$-\frac{dx}{2y} = \sum_{k=0}^{\infty} P_k(\sqrt{1-\lambda}) \left(\frac{2}{\sqrt{1-\lambda}}\right)^k t^{2k+1} \frac{dt}{t}. \quad (12)$$

The coefficients of this differential form on E_λ satisfy ASD congruences, but our interest is in supercongruences for hypergeometric functions ${}_3F_2$, which we obtain using Z.-H. Sun's identity (1.7) [28] (with $x = -\frac{\lambda}{4}$)

$$\sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(-\frac{\lambda}{4}\right)^k = P_n(\sqrt{1-\lambda})^2.$$

With the identity

$${}_3F_2\left(\frac{1}{2}, \frac{1-n}{2}, \frac{1+n}{2}; \lambda\right) = \sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(\frac{-\lambda}{4}\right)^k \quad (13)$$

along with Theorem 7, we obtain the following supercongruences.

Proposition 8. *For CM values λ of the family $E_\lambda : y^2 = (x-1)(x^2 - \frac{1}{1-\lambda})$, such that $\lambda \in \mathbb{Z}_p$ and p is ordinary, for all positive integers m and s with m odd,*

$$\begin{aligned} & {}_3F_2\left(\frac{1}{2}, \frac{1-mp^s}{2}, \frac{1+mp^s}{2}; \lambda\right) \\ & \equiv \left(\left(\frac{\lambda-1}{p}\right) \alpha_{p,\lambda}^2\right) {}_3F_2\left(\frac{1}{2}, \frac{1-mp^{s-1}}{2}, \frac{1+mp^{s-1}}{2}; \lambda\right) \pmod{p^{2s}} \end{aligned}$$

where $\alpha_{p,\lambda}$ is the unit root of $X^2 - [p+1 - \#(E_\lambda/\mathbb{F}_p)]X + p = 0$.

Turning our attention to S_λ and the sequence $F_3(\lambda)_n$, we use the congruence $\binom{\frac{p-1}{2}+k}{2k} \equiv \left(\frac{-1}{16}\right)^k \binom{2k}{k} \pmod{p^2}$ and the equality $\frac{(1/2)_k}{k!} = \binom{2k}{k} \frac{1}{4^k}$ in (13) to obtain

$$F_3(\lambda)_{\frac{p-1}{2}} := {}_3F_2\left(\frac{1}{2}, \frac{1/2}{1}, \frac{1/2}{1}; \lambda\right)_{\frac{p-1}{2}} \equiv P_{\frac{p-1}{2}}(\sqrt{1-\lambda})^2 \pmod{p^2}.$$

Notice that this can also be obtained directly from Clausen's formula expressing certain values of ${}_3F_2$ as the square of values of ${}_2F_1$.

PROOF OF THEOREM 2. We begin by observing that

$$P_{\frac{p-1}{2}}(\sqrt{1-\lambda}) \equiv \begin{cases} 0 & (\text{mod } p) \\ \varepsilon \alpha_{p,\lambda} & (\text{mod } p^2) \end{cases} \quad \begin{array}{l} \text{if } p \text{ is supersingular} \\ \text{if } p \text{ is ordinary} \end{array}$$

where ε is a fourth root of unity. The congruence in the supersingular case is just the first instance of the standard ASD congruences. Since we are assuming λ is a CM value of E_λ , we have supercongruences at ordinary primes by Theorem 7. By the ASD congruences, we may conclude that ε is the fourth root of unity in \mathbb{Q}_p , or possibly in its unramified quadratic extension, that is congruent to $\left(\frac{\sqrt{1-\lambda}}{2}\right)^{\frac{p-1}{2}}$ modulo p . Squaring these congruences for $P_{\frac{p-1}{2}}(\sqrt{1-\lambda})$ immediately yields Theorem 2:

$${}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; \lambda\right)_{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{(\frac{1}{2})_k}{k!}\right)^3 \lambda^k \equiv \left(\frac{\lambda-1}{p}\right) \alpha_{p,\lambda}^2 \pmod{p^2}.$$

Many of the intermediate statements in this argument involve choosing an embedding of $\sqrt{1-\lambda}$ in \mathbb{Q}_p or its unramified quadratic extension, but this choice does not affect the square of the congruence.

We note that this establishes, modulo p^2 , all cases of Conjecture 5.2 of [29] by Z.W. Sun. These conjectures can be written as

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \left(\frac{\lambda}{64}\right)^k \equiv \begin{cases} \left(\frac{c}{p}\right) (4a^2 - 2p) & (\text{mod } p^2) \\ 0 & (\text{mod } p^2) \end{cases} \quad \begin{array}{l} \text{if } \left(\frac{p}{D}\right) = 1 \text{ where } a^2 + Db^2 = p \\ \text{if } \left(\frac{p}{D}\right) = -1 \end{array},$$

with appropriate choices of $D \in \mathbb{Z}_+$ and character $\left(\frac{c}{p}\right)$. Note that $\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \left(\frac{\lambda}{64}\right)^k = {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; \lambda\right)_{\frac{p-1}{2}}$ via the identity $\frac{(1/2)_k^3}{k!^3} = \binom{2k}{k}^3 \frac{1}{64^k}$. These conjectures address the λ -values $\lambda = -8, 1, -\frac{1}{8}, 4, \frac{1}{4}, 64, \frac{1}{64}, -1$, which are all of the CM values for E_λ over \mathbb{Q} , as verified in [4], with the exception of the degenerate case $\lambda = 1$, for which E_λ is not an elliptic curve. The supercongruence for $\lambda = 1$ was proved by Van Hamme in [32] and by Ono in [22].

If E_λ has CM over $K = \mathbb{Q}(\sqrt{-D})$, then the attached 2-dimensional representation ρ decomposes into 2 Grossencharacters when ρ is restricted to $\text{Gal}(\overline{\mathbb{Q}}/K)$. Then at splitting primes p , which are precisely the ordinary primes of E_λ , the trace of the Frobenius is $A_p = \alpha_p + \beta_p$, where both α_p and β_p are in the ring of integers of the quadratic field K and have the same absolute value \sqrt{p} . In the case that K has class number 1, (all Sun λ values correspond to class number 1 cases), then ideals (α_p) and (β_p) are the two distinct prime ideals above p . That is, $\alpha_p = a + b\sqrt{-D}$ and

$\beta_p = a - b\sqrt{-D} = \frac{p}{\alpha_p}$, where a and b are integers or half integers depending on $p \equiv 1$ or $3 \pmod{4}$, such that $a^2 + b^2 D = p$. Our congruences involve α_p^2 , which is just $a^2 - Db^2 + 2ab\sqrt{-D}$. Using $\beta_p^2 = a^2 - Db^2 - 2ab\sqrt{-D} \equiv 0 \pmod{p^2}$ and $a^2 + b^2 D = p$, we have $\alpha_p^2 \equiv 4a^2 - 2p \pmod{p^2}$, which, along with the character $\left(\frac{1-\lambda}{p}\right)$, is the target of Z.-W. Sun's congruences. In the supersingular case, we simply have $\alpha_p = 0$.

Alternately, we note that Ono has explicitly identified the values α_p , for all CM curves E_λ with $\lambda \in \mathbb{Z}$, in Theorem 6 of [22]. These values α_p determine the formal group structure and the ASD congruences (i.e., that $a_p \equiv \left(\frac{1-\lambda}{p}\right) \alpha_p^2 \pmod{p}$); combining this with Coster and Van Hamme's supercongruences gives another proof of Sun's conjectures, that $a_p \equiv \left(\frac{1-\lambda}{p}\right) \alpha_p^2 \pmod{p^2}$.

Note that we have established just the first interesting supercongruence for the truncated hypergeometric functions ${}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda\right)_{\frac{k-1}{2}}$, when $k = p$; we have congruence modulo p^2 where the standard ASD congruences only guarantee congruence modulo p . However, when $\lambda \in \mathbb{Q}$ is a CM value, these congruences actually appear to hold modulo p^3 for ordinary primes p . Further, we expect the whole infinite family of supercongruences to hold for ${}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda\right)_{\frac{p-1}{2}}$:

Conjecture 9. *For CM values λ of the family $E_\lambda : y^2 = (x-1)(x^2 - \frac{1}{1-\lambda})$, such that $\lambda \in \mathbb{Z}_p$ and p is ordinary, for all positive integers m and s with m odd,*

$$\begin{aligned} & {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda\right)_{\frac{mp^s-1}{2}} \\ & \equiv \left(\left(\frac{\lambda-1}{p}\right) \alpha_{p,\lambda}^2\right) {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda\right)_{\frac{mp^s-1-1}{2}} \pmod{p^{2s}} \end{aligned}$$

where $\alpha_{p,\lambda}$ is the unit root of $X^2 - [p+1 - \#(E_\lambda/\mathbb{F}_p)]X + p = 0$.

4. Corollaries

An idea of Gessel for dealing with the supercongruences of the Apéry numbers

$$c_n = {}_4F_3\left(\begin{matrix} -n, -n, 1+n, 1+n \\ 1, 1, 1 \end{matrix}; 1\right) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

is as follows. He identified the auxiliary sequence $b_n = 2 \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 (H_{n+k} - H_{n-k})$, where H_k is the harmonic sum $\sum_{j=1}^k \frac{1}{j}$, and showed that $c_{k+pn} \equiv (c_k + pn b_k) c_n \pmod{p^2}$ where $0 \leq k < p$ [13]. Using the idea of Ishikawa [14], we take $k = n = \frac{p-1}{2}$. It follows that when $c_{(p-1)/2} \not\equiv 0 \pmod{p}$, we have the supercongruence $c_{(p^2-1)/2} \equiv$

$c_{(p-1)/2}^2 \pmod{p^2}$, since $b_{(p-1)/2} \equiv 0 \pmod{p}$ from the p -adic properties of Harmonic sums. In [2], Ahlgren and Ono also need an entity similar to $b_{(p-1)/2}$ to be zero modulo p , which they established using a binomial coefficient identity proved by the WZ method [3].

In the above examples, supercongruences of a sequence c_n were shown to be equivalent to congruences of an auxiliary sequence b_n ; and the congruences for b_n were proved using whatever method applied in each case. Similarly, the supercongruence in Theorem 2 for the sequence $a_n = \sum_{i=0}^n \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i$ is equivalent to the auxiliary congruence in Corollary 3 for the sequence $b_n = \sum_{i=0}^n \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i (6(H_{2i} - H_i) + \frac{(\lambda/64)^{p-1}-1}{p})$. However, we proved our supercongruence using the theorem of Coster and Van Hamme, and thus obtain our auxiliary congruence. We know of no direct proof of Corollary 3; we expect a proof for each fixed individual λ might require some combinatorial identity and additional intelligent guesses of WZ pairs to prove the identity, see [1, 2].

Lemma 10. *For the sequence $a_n = \sum_{i=0}^n \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i$, we introduce the auxiliary sequence $b_n = \sum_{i=0}^n \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i (6(H_{2i} - H_i) + \frac{(\lambda/64)^{p-1}-1}{p})$. Then for any prime p , any k with $\frac{p-1}{2} \leq k < p$, and any n ,*

$$a_{k+pn} \equiv a_k a_n + p b_k \sum_{i=0}^n i \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \pmod{p^2}.$$

PROOF. Notice we can write $a_{k+pn} - a_k a_n$ as the telescoping sum $\sum_{i=1}^n T_{k,i}$, where

$$\begin{aligned} T_{k,n} &= (a_{k+pn} - a_k a_n) - (a_{k+p(n-1)} - a_k a_{n-1}) \\ &= (a_{k+pn} - a_{k+p(n-1)}) - a_k (a_n - a_{n-1}) \\ &= \sum_{i=-p+k+1}^k \binom{2i+2pn}{i+pn}^3 \left(\frac{\lambda}{64}\right)^{i+pn} - \left(\sum_{i=0}^k \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \right) \binom{2n}{n}^3 \left(\frac{\lambda}{64}\right)^n \end{aligned}$$

Using the condition that $\frac{p-1}{2} \leq k < p$, we notice that $\binom{2i+2pn}{i+pn} \equiv 0 \pmod{p}$ if $-p+k+1 < i < 0$. Simplifying modulo p^2 , these terms disappear and we can factor.

$$T_{k,n} \equiv \sum_{i=0}^k \left(\binom{2i+2pn}{i+pn}^3 \left(\frac{\lambda}{64}\right)^{pn} - \binom{2n}{n}^3 \left(\frac{\lambda}{64}\right)^n \binom{2i}{i}^3 \right) \left(\frac{\lambda}{64}\right)^i \pmod{p^2}$$

The factor $\binom{2i+2pn}{i+pn}^3$ may be rewritten as $\frac{-\Gamma_p(1+2i+2pn)^3}{\Gamma_p(1+i+pn)^6} \binom{2n}{n}^3$, where Γ_p is the p -adic gamma function (see Chapter 11 [22]). Let $T_{k,n} \equiv \left(\frac{\lambda}{64}\right)^n \binom{2n}{n}^3 U_{k,n} \pmod{p^2}$, where

$$U_{k,n} = \sum_{i=0}^k \left(\left(\frac{-\Gamma_p(1+2i+2pn)^3}{\Gamma_p(1+i+pn)^6} \right) \left(\frac{\lambda}{64}\right)^{(p-1)n} - \binom{2i}{i}^3 \right) \left(\frac{\lambda}{64}\right)^i.$$

To simplify the p -adic gamma function modulo p^2 , we expand Γ_p in terms of factorials and harmonic sums $H_n = \sum_{i=1}^n \frac{1}{i}$. (By convention, $H_0 = 0$.) We also use the congruence, for $p > 3$, that $H_{p-1} \equiv 0 \pmod{p}$. (Wolstenholme has shown this congruence holds modulo p^2 , though we only need modulo p .)

$$\begin{aligned}\Gamma_p(1+i+pn)^r &\equiv (-1)^{(1+i+pn)r} i!^r (1+pnrH_i) \prod_{j=0}^{n-1} (p-1)!^r (1+pjrH_{p-1}) \pmod{p^2} \\ &\equiv (-1)^{(1+i+pn)r} i!^r (1+pnrH_i) (-1)^{nr} \pmod{p^2} \\ &\equiv (-1)^{(1+i)r} i!^r (1+pnrH_i) \pmod{p^2}\end{aligned}$$

Plugging this into $U_{k,n}$, we have

$$\begin{aligned}U_{k,n} &\equiv \sum_{i=0}^k \left(\left(\frac{(2i)!^3 (1+6pnH_{2i})}{(i)!^6 (1+6pnH_i)} \right) \left(\frac{\lambda}{64} \right)^{(p-1)n} - \binom{2i}{i}^3 \right) \left(\frac{\lambda}{64} \right)^i \pmod{p^2} \\ &\equiv \sum_{i=0}^k \binom{2i}{i}^3 \left(\frac{\lambda}{64} \right)^i \left((1+6pn(H_{2i}-H_i)) \left(\frac{\lambda}{64} \right)^{(p-1)n} - 1 \right) \pmod{p^2}\end{aligned}$$

$$\text{Using } \left(\frac{\lambda}{64} \right)^{(p-1)n} = \left(1 + p \left(\frac{\left(\frac{\lambda}{64} \right)^{p-1} - 1}{p} \right) \right)^n \equiv 1 + pn \left(\frac{\left(\frac{\lambda}{64} \right)^{p-1} - 1}{p} \right) \pmod{p^2},$$

$$\begin{aligned}U_{k,n} &\equiv \sum_{i=0}^k \binom{2i}{i}^3 \left(\frac{\lambda}{64} \right)^i \left((1+6pn(H_{2i}-H_i)) \left(1 + pn \left(\frac{\left(\frac{\lambda}{64} \right)^{p-1} - 1}{p} \right) \right) - 1 \right) \\ &\equiv pn \sum_{i=0}^k \binom{2i}{i}^3 \left(\frac{\lambda}{64} \right)^i \left(6(H_{2i}-H_i) + \left(\frac{\left(\frac{\lambda}{64} \right)^{p-1} - 1}{p} \right) \right) \pmod{p^2}\end{aligned}$$

So $T_{k,n} \equiv pn \binom{2n}{n}^3 \left(\frac{\lambda}{64} \right)^n b_k \pmod{p^2}$. Combining this congruence with the telescoping sum $a_{k+pn} - a_k a_n = \sum_{i=1}^n T_{k,i}$ completes the proof of the lemma.

Using this lemma, we show the equivalence of Theorem 2 and Corollary 3.

PROOF OF COROLLARY 3. We consider $T_{k,n}$ with $k = \frac{p-1}{2}$ and $n = 1$. By definition, $T_{\frac{p-1}{2},1} = a_{\frac{3p-1}{2}} - a_{\frac{p-1}{2}} a_{\frac{3-1}{2}}$; we can rewrite this, modulo p^2 , as $P_{\frac{3p-1}{2}}(\sqrt{1-\lambda})^2 - P_{\frac{p-1}{2}}(\sqrt{1-\lambda})^2 P_{\frac{3-1}{2}}(\sqrt{1-\lambda})^2$. Since the sequence $P_{\frac{n-1}{2}}(\sqrt{1-\lambda})$ satisfies ASD congruences, we know that $T_{\frac{p-1}{2},1} \equiv 0 \pmod{p}$. However, Theorem 2 is precisely the information we need to conclude that $T_{\frac{p-1}{2},1} \equiv 0 \pmod{p^2}$ whenever λ is a CM value of E_λ that embeds in \mathbb{Z}_p .

Thus, since

$$T_{\frac{p-1}{2},1} \equiv \frac{p\lambda}{8} \sum_{i=0}^{(p-1)/2} \binom{2i}{i}^3 \left(\frac{\lambda}{64}\right)^i \left(6(H_{2i} - H_i) + \left(\frac{\left(\frac{\lambda}{64}\right)^{p-1} - 1}{p}\right)\right) \pmod{p^2},$$

we have the desired congruence $b_{\frac{p-1}{2}} \equiv 0 \pmod{p}$ whenever we have supercongruences for $a_{\frac{p-1}{2}}$.

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